A hole in Miles Mathis' derivation of the derivative

Petar Yotsev
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ABSTRACT

In this paper I go over Miles Mathis' derivation [1] in more detail and put to light a hole in "canceling deltas". I fully agree with his arguments against the current derivation but I believe that the one he provides is incomplete.

Differential mechanics

Since this proof heavily depends on differentials and I assume most people are not used to differentials that don't diminish or do something fancy, I'll go over them first.

Let's say you have a sequence, Δx, that represents distance from 0, hence the Δ: 0, 1, 2, 3, 4, 5, ... Its differential ($\Delta\Delta x$) is the sequence that shows the difference between its terms: 1, 1, 1, 1, 1, 1, ... A differential can be defined in two ways. The most clear and concise way would be to express them with function notation, $\Delta f(\Delta x) = f(\Delta x) - f(\Delta x - 1)$ and $\Delta f(\Delta x) = f(\Delta x + 1) - f(\Delta x)$. The difference being the interval about which the differential gives you information. If you imagine each interval as having a beginning and an end, say 3 and 4, the former will give you information about the interval 3−4 if you plug 4 in and the latter will give you information about the interval 3−4 if you plug 3 in. The difference is minor but it does give you different equations. The former definition is more useful when writing code whereas the latter produces better looking sequences and equations. The rest of the paper uses the latter definition.

Finding differentials is roughly equivalent to differentiation and keeping track of the sum is roughly equivalent to integration as long as you remember that you're not finding the derivative or integral. You're finding the differentials. Both operations follow similar rules to the calculus as most people know it. I'll show the tables of differentials bellow. Miles Mathis shows these tables with raw numbers in his original paper but you can just as easily find the expressions for the differentials.

$\Delta x^1 = \Delta x$
$\Delta\Delta x^1 = 1$

$\Delta x^2 = \Delta x^2$
$\Delta\Delta x^2 = 2\Delta x + 1$
$\Delta\Delta\Delta x^2 = 2$

$\Delta x^3 = \Delta x^3$
$\Delta\Delta x^3 = 3\Delta x^2 + 3x + 1$
$\Delta\Delta\Delta x^3 = 6\Delta x + 6$
$\Delta\Delta\Delta\Delta x^3 = 6$

$\Delta x^4 = \Delta x^4$
$\Delta\Delta x^4 = 4\Delta x^3 + 5\Delta x^2 + 4\Delta x + 1$
$\Delta\Delta\Delta x^4 = 12\Delta x^2 + 24\Delta x + 14$
$\Delta\Delta\Delta\Delta x^4 = 24\Delta x + 36$
$\Delta\Delta\Delta\Delta\Delta x^4 = 24$

$\Delta\Delta\Delta x^2 = 2\Delta\Delta x^1$
$\Delta\Delta\Delta\Delta x^3 = 3\Delta\Delta\Delta x^2$
$\Delta\Delta\Delta\Delta\Delta x^4 = 4\Delta\Delta\Delta\Delta x^3$
$\Delta\Delta\Delta\Delta\Delta\Delta x^5 = 5\Delta\Delta\Delta\Delta\Delta x^4$
$\Delta\Delta\Delta\Delta\Delta\Delta\Delta x^6 = 6\Delta\Delta\Delta\Delta\Delta\Delta x^5$
...

Notice that we get some complex differentials from the simple sequences we start with (e.g. $3\Delta x^2 + 3x + 1$ from Δx^3). If you read the table up rather than down, you'll notice that you always start with a constant differential and "integrate" with a specific $+c$ or initial term until you reach a very simple expression. The initial terms for the differentials of Δx^4 are 24, 36, 14, 1 and 0 and it can be said thatΔx^4 is defined by them. This sudden convergence on a simple expression is not always the case though. Most of the time when you integrate, you don't reach a simpler expression than what you started with. For example, you get Pascal's triangle if you integrate with initial terms of 1, 0, 0, 0, ...

1
$(\Delta x + 0)$
$(1/2)(\Delta x + 1)(\Delta x + 0)$
$(1/6)(\Delta x + 2)(\Delta x + 1)(\Delta x + 0)$
$(1/24)(\Delta x + 3)(\Delta x + 2)(\Delta x + 1)(\Delta x + 0)$
...

1	1	1	1	1	1	1	1	1	1	1	...
0	1	2	3	4	5	6	7	8	9	10	...
0	1	3	6	10	15	21	28	36	45	55	...
0	1	4	10	20	35	56	84	120	165	220	...
0	1	5	15	35	70	126	210	330	495	715	...
0	1	6	21	56	126	252	462	792	1287	2002	...

This point about initial terms will be important later on so keep it in mind. Also of note is the fact that differentials of multiple sequences can be equivalent without the sequences being equivalent. To see what I mean let's look at the differential bellow any constant sequence − 0. All constant differentials share this differential but, obviously, that does not make them equivalent.

The derivation

We are given $\Delta t = \Delta x^3$
We find from the table $3\Delta\Delta\Delta x^2 = \Delta\Delta\Delta\Delta x^3$
We simplify $3\Delta x^2 = \Delta\Delta x^3$
We seek $\Delta\Delta t$
We notice $\Delta\Delta t = \Delta\Delta x^3$ since we can always add a delta to both sides*
We substitute $\Delta\Delta t = 3\Delta x^2$
$\Delta\Delta t = \Delta t'$
So $\Delta t' = 3\Delta x^2$

Throughout the paper, Miles Mathis talks about "canceling", "adding" and "multiplying by" deltas. He justifies this by the fact that the same underlying variable − Δx − is on both sides of the equalities but he never fully explains what it means. To add deltas is to differentiate (go down the table) and to cancel deltas is to integrate (go up the table) using a known initial term (as is the case with Δx^n sequences) or one of your choosing (as in the example with Pascal's triangle). You can't talk about the Δ separately from x just like you can't talk about the little line that goes through f separately from f, it's part of the name of the sequence or differential.

With that out of the way, let's go to the second step. We do indeed find that equality from the tables but we also find $\Delta\Delta\Delta\Delta x^3 = (1/4)\Delta\Delta\Delta\Delta\Delta x^4$ and I would like to use that instead. Next we simplify and substitute $\Delta\Delta x^3 = (1/4)\Delta\Delta\Delta x^4 = 3\Delta x^2 + 6\Delta x + 7/2$. We seem to have brought along an extra $6\Delta x + 7/2$. Let's try finding another equality from the table, $\Delta\Delta\Delta\Delta x^3 = \Delta\Delta\Delta\Delta x^3$. Simplify and substitute $\Delta\Delta x^3 = 3\Delta x^2 + 3x + 1$. Again with the extra baggage. The differential equalities are equalities between differentials, as in the whole expression. Δx^2 simply happens to share the same name as its expression. That's why the "substitute" step is missing. It's because the name given to the differential simply happens to match its expression.

It seems that I can make $\Delta\Delta x^3$ equal whatever I want using this method. The reason for this is that "simplifying" really means integrating. In the derivation, we have some sequence that we differentiate ($\Delta x^3 \to \Delta\Delta\Delta\Delta x^3$) and get rid of the initial terms that define it in the process. We then switch to differentials from some other sequence ($\Delta x^3 \to 3\Delta x^2$) via the differential equalities and integrate using **the other sequence's initial terms**. As I showed earlier with the 0 differential, equalities between differentials do not imply equalities between their integrations, so this step is illegal. Although the integration itself is valid since the initial terms are known and clearly defined, the choice of $3\Delta x^2$ is unjustified outside of historical bias and because it gets us to the right answer.

Apart from this hole, I believe the rest of the reasoning and derivation is sound and if a concrete reason is given for specifically switching to the differentials of $3\Delta x^2$, I'll be perfectly happy. Sadly, I can't think of such a reason.

References

1. Miles Mathis, *A Re-definition of the Derivative (why the calculus works−and why it doesn't)* (2004). http://milesmathis.com/are.html.

Updated − May 6, 2021 − The differential expressions are in fact true for different values of $\Delta\Delta x$ so I removed the part saying they're not. I also fixed the margins.