The death of critical evaluation

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ABSTRACT

We don't have perfect knowledge, therefore there are errors in our theory. The only way to root out these errors is by looking at the derivation of our theory. In this paper I explain in detail how this point is missed and provide an example with the definition of the derivative and subsequently the limit.

The kind of mathematics that are thought in the educational institutions are shadows of the real thing. You are thought methods for solving problems and are given problems which those methods solve. You are supposed to map information to you given in a question to the variables in formulas you study, maybe rearrange a bit and get the output. I've repeatedly heard teachers say "if you don't get it, just work back from the formulas" but that's backwards. The formulas and equations are mere links to the systems they describe. It is the underlying mechanics of the relationships described by the equations that you are supposed to deal with. Once you can imagine a systems, its mechanics and how the variables interact with them, all else is arbitrary substitution and manipulation. Real mathematics aren't about the arbitrary substitutions and manipulations. They are about creating or examining systems from known mechanics that describe something, be it in the real world or not. Don't get me wrong, it's good to be skilled in manipulations but that skill can easily be acquired as a consequence of creating or examining systems. This is because good manipulation is *required* for such tasks.

Perhaps this is still a little too abstract so I'll give you an example of a wild goose chase I went on while searching for mechanics where mainly manipulations were provided. Let's look at the definition of the derivative from a textbook published by Cambridge [1]:

> Near any particular point, P, the value of the function changes by an amount Δf, say, as x changes by a small amount Δx. The slope of the tangent to the graph of $f(x)$ at P is then approximately $\Delta f/\Delta x$, and the change in the value of the function is $\Delta f = f(x+\Delta x) - f(x)$. In order to calculate the true value of the gradient, or first derivative, of the function at P, we must let Δx become infinitesimally small. We therefore define the first derivative of $f(x)$ as
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> $$f'(x) = \frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{2.1}$$
>
> provided that the limit exists. The limit will depend in almost all cases on the value of x. If the limit does exist at a point $x = a$ then the function is said to be differentiable at a; otherwise it is said to be non-differentiable at a. The formal concept of a limit and its existence or non-existence is discussed in chapter 4; for present purposes we will adopt an intuitive approach.

Notice that the following function

$$g(\Delta x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

is undefined for $\Delta x = 0$. We need to take its limit to bring it back to reality. This is a different function from $f(x)$ which is defined at a. $f'(x)$ can be said to be defined in terms of $\lim_{\Delta x \to 0} g(\Delta x)$. So how do we evaluate $\lim_{\Delta x \to 0} g(\Delta x)$? Well, let's look at chapter 4.7 "Evaluation of limits" where we will take a look at the formal definition [2]:

The idea of the limit of a function $f(x)$ as x approaches a v alue a is fairly intuitive though a strict definition exists and is state below. In many cases the limit of the function as x approaches a will be simply the value $f(a)$, but sometimes this is not so. Firstly, the function may be undefined at $x = a$, as, for example, when

$$f(x) = \frac{sin(x)}{x},$$

which takes the value 0/0 at $x = 0$. However, the limit as x approaches zero does exist and can be evaluated as unity using l'Hôpital's rule below. ... The strict definition of a limit is that if $\lim_{x \to a} f(x) = l$ then for any number ϵ however small, it must be possible to find a number η such that $|f(x) - l| < \epsilon$ whenever $|x - a| < \eta$. In other words, as x becomes arbitrarily close to a, $f(x)$ becomes arbitrarily close to its limit, l. To remove any ambiguity, it should be stated that, in general, the number η will depend on both ϵ and the form of $f(x)$.

That's pretty cryptic so let's look at a specific case to breakdown what this strict definition really means. Let's say $f(x) = x^3$ and$a = 0$ so $|x - a|$ will start at, say , 1 and go to 0. We are closing the distance from 1 to 0 and that distance must be less than η. On each step that that distance becomes smaller, we will evaluate the distance from $f(x)$ to l. l being $f(a) = 0^3 = 0$ in this case and $f(x)$ at the beginning being $f(1) = 1^3 = 1$. So we have another distance that goes from 1 to 0 and that distance must be less than ϵ in every step $|x - a|$ is less than η.

I find this very strange because l is supposed to be the evaluated limit. That means that you don't know l when calculating $|f(x) - l|$. In my case above, $f(a)$ did exist so I could calculate it but remember, the limit is supposed to be used where $f(a)$ is undefined. At the start of the definition it says that **if** $\lim_{x \to a} f(x) = l$ is true, you can find η, but how can you know if it is true or not if you can't find l in the first place? You simply can't know the equation is true so you can't find η given a very small ϵ.

What exactly is η and ϵ anyway? Well, we aren't told in this chapter. There's not even a reference, be it within the book or otherwise. We are only told that η somehow magically depends on ϵ and $f(x)$. Basically, this "strict" and "formal" definition isn't telling us anything. But they did state that you can evaluate $\lim_{x \to a} f(x)$ where $f(a)$ is undefined using l'H ôpital's rule just above the definition. Let's look at that now and return to η and ϵ if we find the definition of a limit in l'Hôpital's rule.

(iii)(c) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided that the numerator and denominator are not both equal to zero or infinity.

(v) L'hôpital's rule may be used; it is an extension of (iii)(c) above. In cases where both numerator and denominator are zero or both are infinite, further consideration of the limit must follow. Let us consider $\lim_{x \to a} f(x)/g(x)$, where $f(a) = g(a) = 0$. Expanding the numerator and denominator as Taylor series we obtain

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$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

This rule applied to the numerator and denominator of $\lim_{\Delta x \to 0} g(\Delta x)$ looks like it solves all of our problems, except for the fact that it's defined in terms of the first derivative − the thing we were trying to find in the first place. The definition is circular, therefore $f'(x)$, being defined in terms of $\lim_{\Delta x \to 0} g(\Delta x)$, is also undefined; not only at point P but also at any other point, since you need to go through $\lim_{\Delta x \to 0} g(\Delta x)$ to find $f'(x)$ for any x.

What this means is that when you differentiate using this method for a specific $f(x)$, say, $f(x) = x^2$, you can't go from $f'(x) = 2x + \Delta x$ to $f'(x) = 2x$. The definition of $f'(x)$ that we went o ver doesn't have the mechanics to justify that manipulation. Does this mean that the derivative isn't $2x$? No, it means $f'(x)$ as defined above isn't the derivative for the same reason that a definite integral can never become an indefinite integral or that a length can't become a point with a scale factor different from 0. The Δx specifies the interval of $f'(x)$ for a particular x. Getting rid of it because it' s arbitrarily small compared to the values you plug in $f(x)$ in a specific problem is not a slight error(/correction), it brings the whole of $f'(x)$ into the undefined that is $g(0)$. Again, I'm not saying that the derivative isn't $2x$ or that $dy/dx \neq nx^{n-1}$. I'm saying that this method of getting to $2x$ is fundamentally flawed and that there exists perhaps another method to get to the derivative as we know it that actually works. Miles Mathis claims to have such a method that doesn't use limits [3]. Or there might be a definition of the limit that works. This particular solution just isn't it.

You might say "you didn't prove the definition of the derivative is wrong, you proved the textbook was in error" but you'd be missing the point. They bear the burden of proof and they have failed to prove it. The sane action would be to conclude that the definition is wrong. Throughout this whole ordeal we are told how to work with given tools, not how those tools work: manipulations instead of mechanics. Sure, we are given definitions but they are never meant to be scrutinized. We are meant to and actively encouraged to follow the intuitive way of thinking, this despite the fact we are also actively going against intuitions such as "devision by zero is undefined". You can't eat your cake and have it too.

Critical evaluation is not rejection on first sight but it is a rejection of a certain way of thinking "this is surely defined rigorously somewhere else" and going on to memorize manipulations. I admit I am guilty of suspending my disbelieve where I shouldn't have and not being critical of presented arguments. I consider it my greatest error and regret. But I didn't recognize it until something I though must "surely be defined rigorously somewhere else" turned out to be backed up by induction instead of deduction. That's why I included the example above: to hopefully open you up to the reality of errors in common theory that you're thought so that you evaluate it critically.

Open whatever textbook you have on mathematics and see how much of the manipulations given to you are backed up by a derivation or have a reference to one. You'd be surprised by how much is simply stated as true which is not at all obvious or even contradictory with previous statements. If a derivation depends on something defined two chapters later, you might want to do a double take. This is the fundamental problem I have with the way things are thought. You are rarely shown where equations come from and are expected to blindly trust that they are true. If you study physics, you aren't taken to the Principia when studying mechanics and you don't look at Einstein's original papers before using $E = mc^2$. In the UK, where I was first thought calculus and differentiation, the limit wasn't even in the specification for the course.

References

1. K. F. Riley, M. P. Hobson, and S. J. Bence, “Differentiation from first principles” in *Mathematical methods for physics and engineering,* p. 42, Cambridge University Press, Cambridge (2006).
2. K. F. Riley, M. P. Hobson, and S. J. Bence, “Evaluation of limits” in *Mathematical methods for physics and engineering,* p. 141, Cambridge University Press, Cambridge (2006).
3. Miles Mathis, *A Re-definition of the Derivative (why the calculus works−and why it doesn't)* (2004). http://milesmathis.com/are.html.

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